Reply to Melrose's comment on 'On the multifractal nature of fully developed turbulence and chaotic systems'

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COMMENT

Reply to Melrose's comment on 'On the multifractal nature of fully developed turbulence and chaotic systems'

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Abstract. We reply to the comments made by Melrose on our earlier paper.

The comment of Melrose (1986) (henceforth referred to as M) points out an interesting feature of the random β model which is worth stressing and actually was not in our original paper (Benzi *et al* 1984). Nevertheless some conclusions of the comment are not correct and need a clarifying answer.

In the following we limit the discussion to case (a) of the comment where the decoration for each eddy is chosen independently from some distribution. In fact case (b) of the comment was never used by us because it was unphysical.

We have indeed emphasised in § 4 devoted to chaotic attractors that, at each step i of the fragmentation, 'the β_i depend on their particular hypercube father'.

The misunderstanding might be due to the notation: we indicated in (4.6) with α the particular history which produces a hypercube but for a given configuration there are as many histories as hypercubes, of course.

The main claim of the comment is that the fractal built up in case (a) is a homogeneous fractal since the moments of the number N_n of active eddies on scale l_n are related to the first moment by a power law

$$\langle N_n^q \rangle \propto \langle N_n \rangle^q \sim l_n^{-D_F q}. \tag{1}$$

Although (1) is correct other quantities may have non-trivial behaviours; let us consider the fluctuations of the velocity:

$$\langle \delta v(l_n)^p \rangle \sim l_n^{\zeta_p} \qquad \text{for } l_n \to 0. \tag{2}$$

It is indeed easy to show that ζ_p is now a non-linear function of p. We want here to stress that the term 'inhomogeneous' in our paper (henceforce referred to as I) must be understood as being related to the distribution of 'mass' (in turbulence the density of energy dissipation, in a chaotic system the density of points generated by the evolution in the phase space) and not to 'geometrical' properties, so that no contradiction is present.

Let us now show that the result (6) of M can be derived with simple computations, without leading to any contradiction. In particular the comment wrongly states (see (5) of M) that

$$\langle N_n^q \rangle = \langle N_1^q \rangle^n \qquad \text{for } q \neq 1 \tag{3}$$

follows in I from the independence of the random variables $\beta_i(k)$.

Now the number of eddies is given in (3.3) of I:

$$N_n = 2^{3n} \prod_{i=1}^{n} B_i \tag{4}$$

where

$$B_i = \frac{1}{N_i} \sum_{k=1}^{N_i} \beta_i(k)$$

and the independence of the β does not imply that of the B.

As remarked in M, this is due to the coupling among steps: the mean value at step n depends on the mean value at step $n-1$. In fact we can write the number of eddies as

$$N_n = N_{n-1} 2^3 \left(\frac{1}{N_{n-1}} \sum_{k=1}^{N_{n-1}} \beta_n(k) \right). \tag{5}$$

Averaging (5) and using the independence of the β one has:

$$\langle N_n \rangle = \langle N_{n-1} \rangle \cdot 2^3 \{\beta\} \tag{6}$$

where $\{\cdot\}$ indicates the average of the probability distribution of the β. The result

$$\langle N_n \rangle = \langle N_1 \rangle^n \tag{7}$$

follows by iterating (6).

On the other hand, it is easy to see that

$$\langle N_n^2 \rangle = 2^6 \left\langle \sum_{k,l=1}^{N_{n-1}} \{\beta_n(k)\beta_n(l)\} \right\rangle. \tag{8}$$

The sum $\Sigma_{k,l}$ contains N_{n-1} terms $\{\beta^2\}$ (i.e. terms for which $k=l$) and $(N_{n-1}^2 - N_{n-1})$ terms $\{\beta\}^2$ (i.e. $k \neq l$). In the limit of large n (i.e. $N_{n-1} \gg 1$) we obtain:

$$\langle N_n^2 \rangle \propto (2^3\{\beta\})^{2n}. \tag{9}$$

The generalisation of (9) to a generic value of q is straightforward.

The result (1) is simply due to the fact that for $i \gg 1$

$$B_i = \{\beta\}(1 + O(1/\sqrt{N_i})).$$

On the contrary the coupling among steps is absent (in the hypothesis of the independence of the β) when the moments of the δv are computed because the $\Pi_{i=1}^n \beta_i$ (and not the $\Pi_{i=1}^n B_i$) is involved.

References

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